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Universal Amplitude Ratios in the 3D Ising Model

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Abstract

We present a high precision Monte Carlo study of various universal amplitude ratios of the three dimensional Ising spin model. Using state of the art simulation techniques we studied the model close to criticality in both phases. Great care was taken to control systematic errors due to finite size effects and correction to scaling terms. We obtain $C_+/C_- = 4.75(3)$, $f_{+,2nd}/f_{-,2nd} = 1.95(2)$ and $u^* = 14.3(1)$. Our results are compatible with those obtained by field theoretic methods applied to the ϕ^4 theory and high and low temperature series expansions of the Ising model. The mismatch with a previous Montecarlo study by Ruge et al. remains to be understood.

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1 Introduction

In the neighbourhood of a second order phase transition various quantities display a singular behaviour. In this limit most of the microscopic features which characterize a given model become irrelevant and models which differ at the microscopic level may share the same singular behaviour. This is the basis of the concept of universality. The first, well known, consequence of universality is that different models belonging to the same universality class share the same critical indices. However the hypothesis of universality of the various scaling functions has much stronger implications and it is possible to show that models belonging to the same universality class are also characterized by the same values of some critical-point amplitude combinations [1]. Let us see a simple example. Near the critical point the correlation length ξ diverges as

$$\begin{aligned} \xi &\sim f_+ \; t^{-\nu}; \qquad t > 0 \\ \xi &\sim f_- \; (-t)^{-\nu}; \;\; t < 0 \end{aligned} \tag{1}$$

with

$$t = \frac{T - T_c}{T_c} \tag{2}$$

where T is the temperature and T_c the critical temperature. Different models in the same universality class share not only the same critical exponent ν, but also the same dimensionless combination of critical amplitudes f_+/f_-. This is particularly relevant from the experimental point of view, since in general critical amplitudes are more easily detectable than critical indices and allow a simpler identification of the universality class. In fact the variations of the critical indices between different universality classes are in general rather small, while the amplitude ratios may vary by large amounts. In this paper we shall be in particular interested in the universality class of the three dimensional Ising model which has several interesting experimental realizations, ranging from the binary mixtures to the liquid vapor transitions.

The two standard approaches to the evaluation of these amplitudes ratios in the Ising case are the use of field theoretic methods applied to the ϕ^4 theory [2]-[8], and the extrapolation to criticality of low and high temperature series expansions on various lattices [9]. All these estimates are in general in rather good agreement among them (for a comparison and a discussion see sect.5).

In order to obtain results from Montecarlo simulations relevant for the scaling limit we have to control both finite size effects as well as corrections to scaling. This means that the linear lattice sizes L have to be chosen such that $L >> \xi$ while $\xi \to \infty$ as $\beta \to \beta_c$. In practice one has to carefully check which factor of L/ξ is required to obtain results sufficiently close to the thermodynamic limit. While in the high temperature phase $L/\xi \approx 7$ turns out to be sufficient to give thermodynamic limit results within numerical accuracy, in the low temperature phase this factor has to be doubled at least. The value of ξ that can be reached, and hence the

control of corrections to scaling, is limited by the CPU time available for the study. In the present paper the largest correlation length is $\xi = 11.884(9)$ in the high temperature phase and $\xi = 6.208(18)$ in the low temperature phase.

One also should note that simulations in the low temperature phase are considerably more difficult than those in the high temperature phase of the model. In the low temperature phase conceptual as well as practical problems caused by spontaneous symmetry breaking arise. Furthermore the determination of the correlation length is complicated by the occurrence of secondary correlation lengths which are close to the leading one.

For all these reasons only in these last years there have been some attempts to measure these ratios in Montecarlo simulations [11]. However the results are rather puzzling. For instance, in the case of the ratio f_+/f_- discussed above, the Montecarlo estimate, which is $f_+/f_- = 2.06(1)$ [11], disagrees with the one obtained with strong/weak coupling series $f_+/f_- = 1.96(1)$ [9], while the field theoretical estimate $f_+/f_- = 2.013(28)$ [8] lies in between the two.

The aim of our work is to show that Montecarlo estimates of the amplitude ratios can indeed be competitive with other approaches. To this end we have elaborated a technique to directly extract the various amplitude ratios, without evaluating the single amplitudes thus avoiding all the uncertainties related to the critical indices. We shall discuss this point in sect.5 below. Besides this, we have devoted a great care throughout the paper to keep under control systematic errors due to finite size effects and corrections to scaling. Finally we have used state of art simulation techniques to obtain high precision estimates of the observables near the critical point, in both phases. The simulations in the high temperature phase have been performed using Wolff's single cluster algorithm. Here the improved estimators give a great boost to the accuracy of the results. However in the low temperature phase, due to the finite magnetization, the improved estimators of the cluster-algorithm are of little help. Hence we simulated here with a multispin coding implemented Metropolis-like algorithm.

As we shall see our results are comparable in precision and agree with the most recent field-theoretic and strong/weak coupling estimates, while they are incompatible with the MC results of Ruge et al. [11]. A more detailed comparison of the data might be helpful to understand this discrepancy.

This paper is organized as follows: In sect.2 we have collected some informations on the three dimensional Ising model and on the observables that we study. In sect.3 we discuss the details of the simulation, while in sect.4 we analyse the scaling behaviour of the measured quantities: magnetization, susceptibility and correlation lengths. In sect.5 we study the amplitude ratios and compare our results with other existing estimates and with the experiments. Finally sect.6 is devoted to some concluding remark.

2 General Setting

2.1 The Model

We study the Ising spin model in three dimensions on a simple cubic lattice. The action is given by

$$S_{spin} = -\beta \sum_{<n,m>} s_n s_m \ , \tag{3}$$

where the field variable s_n takes the values -1 and $+1$; $n \equiv (n_0, n_1, n_2)$ labels the sites of the lattice and the notation $< n, m >$ indicates that the sum is taken on nearest neighbour sites only. The coupling β is defined as $\beta \equiv \frac{1}{kT}$, hence the reduced temperature t can be written as

$$t = \frac{\beta_c - \beta}{\beta} \quad , \tag{4}$$

where $\beta_c \equiv \frac{1}{kT_c}$. We shall consider in the following n_1 and n_2 as "space" directions and n_0 as the "time" direction and shall sometimes denote the time coordinate n_0 with τ. We always consider lattices of equal extension L and periodic boundary conditions in all three directions.

2.2 The observables

2.2.1 Magnetisation

The magnetization of a given configuration is defined as:

$$m = \frac{1}{V} \sum_i s_i \quad , \tag{5}$$

where $V \equiv L^3$ is the volume of the lattice. However, in a finite volume the Z_2 symmetry of the model can not be broken for any nonzero temperature. Hence the expectation value of m vanishes.

In order to obtain the magnetization of the model in the low temperature phase one should add a magnetic field h in order to break the symmetry. Then one should first take the thermodynamic limit at finite magnetic field and then take the limit of vanishing magnetic field. However it is difficult to follow this route in a numerical study.

As an alternative Binder and Rauch [12] suggested to simulate the finite lattices at vanishing external field and study the quantity

$$< m > \ \equiv \ \lim_{L \to \infty} \sqrt{< m^2 >} \tag{6}$$

However it turns out that this is not the best choice. In fact this observable is affected by strong finite size effects [13] which would require very large lattices to

obtain reliable estimates of the infinite volume magnetization. It has been recently observed [14] that a much more stable observable is:

$$< m > \;\; \equiv \;\; \lim_{L\to\infty} < |m| > \quad . \tag{7}$$

The finite size behaviour of this observable, was carefully studied in [14] where it was shown that the asymptotic, infinite volume, value is reached for lattices of size $L >\sim 8\xi$, where ξ denotes the correlation length. In our simulations we always used lattice sizes much larger than this threshold.

Close to the critical temperature, the magnetization is supposed to scale as

$$< m >\sim \;\; B \; (-t)^{\beta}; \;\; t < 0 \tag{8}$$

where the critical exponent β should not be confused with the inverse temperature.

2.2.2 Magnetic susceptibility

The susceptibility gives the response of the magnetization to an external magnetic field.

$$\chi = \frac{\partial < m >}{\partial H} \tag{9}$$

One easily derives that the magnetic susceptibility can be expressed in terms of moments of the magnetization as follows:

$$\chi = V \left(< m^2 > - < m >^2 \right) \qquad . \tag{10}$$

Close to the critical temperature the magnetic susceptibility is supposed to scale as

$$\begin{aligned} \chi &\sim C_+ \; t^{-\gamma}; \qquad t > 0 \\ \chi &\sim C_- \; (-t)^{-\gamma}; \;\; t < 0 \qquad . \end{aligned} \tag{11}$$

2.2.3 Exponential correlation length.

We consider the decay of so called time-slice correlation functions. The magnetization of a time slice is given by

$$S_{n_0} = \frac{1}{L^2} \sum_{n_1,n_2} s_{(n_0,n_1,n_2)} \quad . \tag{12}$$

Let us define the correlation function

$$G(\tau) = \sum_{n_0} \left\{ \langle S_{n_0} S_{n_0+\tau} \rangle - \langle S_{n_0} \rangle^2 \right\} \quad . \tag{13}$$

The large distance behaviour of $G(\tau)$ is given by

$$G(\tau) \propto \exp(-\tau/\xi) \quad , \tag{14}$$

where ξ is the exponential correlation length.

Close to criticality the behaviour of the correlation length is governed by the scaling laws

$$\begin{array}{lcll} \xi & \sim & f_+ \ t^{-\nu}; & t > 0 \\ \xi & \sim & f_- \ (-t)^{-\nu}; & t < 0 \end{array} \quad . \tag{15}$$

2.2.4 Second moment correlation length.

The square of the second moment correlation length is defined for a generic value of the spacetime dimensions d by

$$\xi_{2nd}^2 = \frac{\mu_2}{2d\mu_0} \quad , \tag{16}$$

where

$$\mu_0 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} \langle s_m s_n \rangle_c \tag{17}$$

and

$$\mu_2 = \lim_{L\to\infty} \frac{1}{V} \sum_{m,n} (m-n)^2 \langle s_m s_n \rangle_c \ . \tag{18}$$

The connected part of the correlation function is given by

$$\langle s_m s_n \rangle_c = \langle s_m s_n \rangle - \langle s_m \rangle^2 \tag{19}$$

This estimator for the correlation length is very popular since its numerical evaluation (say in Montecarlo simulations) is simpler than that of the exponential correlation length. Moreover it is the length scale which is directly observed in scattering experiments. However it is important to stress that it is not exactly equivalent to the exponential correlation length. The relation between the two can be obtained as follows. Let us write

$$\begin{aligned} \mu_2 &= \frac{1}{V} \sum_{m;n} (n-m)^2 \ \ \langle s_m s_n \rangle_c \\ &= \frac{1}{V} \sum_{n;m} \sum_{\mu=0}^{d-1} (n_\mu - m_\mu)^2 \ \ \langle s_m s_n \rangle_c \\ &= \frac{d}{V} \sum_{n;m} (n_0 - m_0)^2 \ \ \langle s_m s_n \rangle_c \quad . \end{aligned} \tag{20}$$

Due to the exponential decay of the correlation function this sum is certainly convergent and we can commute the spatial summation with the summation over configurations so as to obtain

$$\mu_2 = d \sum_{\tau=-\infty}^{\infty} \tau^2 \ \ \langle S_0 \ S_\tau \rangle_c \tag{21}$$

with S_{n_0} given by eq.(12). Analogously one obtains

$$\mu_0 = \sum_{\tau=-\infty}^{\infty} \langle S_0 \; S_\tau \rangle_c \tag{22}$$

If we now insert these results in eq.(16), assume a multiple exponential decay

$$\langle S_0 \; S_\tau \rangle_c \propto \sum_i \; c_i \; \exp(-|\tau|/\xi_i) \;\; , \tag{23}$$

and replace the summation by an integration over τ we get

$$\xi_{2nd}^2 = \frac{1}{2} \; \frac{\int_{\tau=0}^{\infty} \mathrm{d}\tau \; \tau^2 \; \exp(-\tau/\xi)}{\int_{\tau=0}^{\infty} \mathrm{d}\tau \; \exp(-\tau/\xi)} = \frac{\sum_i c_i \xi_i^3}{\sum_i c_i \xi_i} \; , \tag{24}$$

which is equal to ξ^2 if only one state contributes. An interesting consequence of this analysis is that the difference from one of the ratio ξ/ξ_{2nd} gives an idea of the density of the lowest states of the spectrum. If these are well separated the ratio will be almost one, while a ratio significantly higher than one will indicate a denser distribution of states.

The critical behaviour of ξ_{2nd} is governed by the same critical index ν, so near the critical point we expect:

$$\begin{aligned} \xi_{2nd} \;\; &\sim \;\; f_{+,2nd} \; t^{-\nu}; \qquad t > 0 \\ \xi_{2nd} \;\; &\sim \;\; f_{-,2nd} \, (-t)^{-\nu}; \;\; t < 0 \end{aligned} \tag{25}$$

2.3 Critical indices

Our aim is to obtain high precision estimates for some amplitude ratios. To this end we need to use as input informations the critical temperature β_c and the values of the critical indices [1] defined above. We list in tab.1 some estimates for these quantities, obtained with field theoretical methods, strong coupling series and Montecarlo simulations.

In general these values are in rather good agreement among them, despite the fact that they were obtained with very different methods. As input parameters for our analysis we have decided to choose the following values:

$$\beta_c = 0.2216544(3) \; , \quad \gamma = 1.2390(15) \; , \quad \nu = 0.6310(15) \; , \tag{26}$$

$$\beta = 0.3270(6) \; , \quad \theta = 0.51(3). \tag{27}$$

which are obtained combining together the results of [14] and [16] and satisfy the scaling relations. Let us stress however that our results are only slightly affected by this choice [2].

[1] As a matter of fact, for the actual determination of the amplitude ratios we only need to know β_c and θ. The values of the other critical indices will only be used in comparing our results with those obtained with the series expansions.

[2] The systematic errors due to the uncertainties in the choice of β_c and θ turn out to be much smaller than the statistical fluctuations of our estimates. In any case, both are taken into account in the errors that we quote in our final results.

Table 1: *Results for β_c and for the critical indices given in the literature*

ref.	method	β_c	γ	ν	β	θ
[15]	FT		1.241(4)	0.630(2)	0.324(6)	0.496(3)
[16]	ϵ-expansion		1.2390(25)	0.6310(15)	0.3270(15)	0.51(3)
[17]	$d=3$		1.2405(15)	0.6300(15)	0.3250(15)	0.50(2)
[18]	MCRG	0.221652(3)		0.624(1)		0.50-0.53
[19]	MCRG	0.221655(1)(1)		0.625(1)		0.44
[20]	MC,FS	0.2216546(10)	1.237(2)	0.6301(8)	0.3267(10)	0.52(4)
[21]	MC	0.2216576(22)	1.239(7)	0.6289(8)	0.3258(44)	
[14]	MC	0.2216544(3)			0.3269(6)	0.508(25)
[22]	HT		1.237(2)	0.6300(15)		0.52(3)
[23]	HT		1.239(3)	0.632(3)		0.52(3)
[24]	HT		1.2385(25)	0.6305(15)		0.57(7)
[25]	HT		1.2395(4)	0.632(1)		0.54(5)

2.4 Amplitude ratios

In the following we shall be interested in these scaling functions:

$$\Gamma_\chi(t) \equiv \frac{\chi(t)}{\chi(-t)}\ , \qquad \Gamma_\xi(t) \equiv \frac{\xi_{2nd}(t)}{\xi_{2nd}(-t)} \qquad (t>0), \tag{28}$$

$$u(t) \equiv \frac{3\ \ \chi(t)}{\xi_{2nd}^3(t)m^2(t)} \qquad (t<0), \tag{29}$$

$$\Gamma_c(t) \equiv \frac{\chi(t)}{\xi_{2nd}^3(t)m^2(-t)} \qquad (t>0) \tag{30}$$

(notice the factor of three difference between the definitions of Γ_c and u). While Γ_χ, Γ_ξ and Γ_c mix low and high temperature observables, u only contains quantities evaluated in the broken symmetry phase. Γ_c and u are scale invariant thanks to the following scaling (and hyperscaling) relations among the critical exponents:

$$\alpha + 2\beta + \gamma = 2\ , \qquad d\nu = 2-\alpha \quad . \tag{31}$$

In particular, u plays the important role of a low temperature renormalized coupling constant in the study of the ϕ^4 theory directly in $d=3$.

It is important to notice that Γ_c is related to the ratio of two amplitude combinations (in which $A(t)$ denotes the specific heat):

$$R_c \equiv \frac{\chi(t)\ A(t)}{m^2(-t)} \qquad R_\xi \equiv \xi_{2nd}(t)\ A(t)^{1/3} \qquad (t>0) \tag{32}$$

which have been widely studied in the literature since they can be rather easily evaluated in experiments. The relation is: $\Gamma_c = R_c/R_\xi^3$. In the scaling limit these

functions are related to the amplitudes defined in eq.(8,11,15,25) as follows:

$$\lim_{t\to 0}\ \Gamma_\chi(t) = \frac{C_+}{C_-} \quad , \qquad \lim_{t\to 0}\ \Gamma_\xi(t) = \frac{f_{+,2nd}}{f_{-,2nd}} \quad , \tag{33}$$

$$\lim_{t\to 0}\ u(t) \equiv u^* = \frac{3\ C_-}{f^3_{-,2nd}\ B^2} \quad , \qquad \lim_{t\to 0}\ \Gamma_c(t) = \frac{C_+}{f^3_{+,2nd}\ B^2} \quad . \tag{34}$$

Finally we shall also be interested in evaluating the ratio:

$$\frac{\xi}{\xi_{2nd}} \tag{35}$$

both above and below the critical point.

2.5 Series expansions

A very powerful approach to the study of the three dimensional Ising model is represented by the series expansions which lead to estimates for several quantities near the critical point which are competitive with the most precise Montecarlo simulations. In the following we shall compare our results in the low temperature phase (which is the one in which simulations are more difficult and results are in general affected by stronger finite size effects) with those obtained with series expansions with the twofold aim of testing the reliability of our simulations and of comparing the precision of the two methods. The low temperature regime is also particularly interesting because recently these series have been extended up to very high orders [26, 27, 28]. Some informations on these series can be found in tab.13. In order to extract from the series the estimates for the observables in which we are interested and to quantify the uncertainty of such estimates we use the so called "double biased inhomogeneous differential approximants" (IDA). The technique of IDA is described in [9, 29], to which we refer for notations and further details. Following ref. [9] we use the notation [K/L;M] for the approximants. In order to keep the fluctuations of the results under control, we have chosen to use double biased IDA [9], namely we fix both the critical coupling β_c and the critical index describing the critical behaviour of the observable. As K and L vary we obtain several different IDA's and correspondingly several different estimates of the observable. As we approach the critical point these estimates start to spread out, indicating that we are pushing the series toward its convergence threshold. The last problem is then to extract from this set of values the best estimate and its uncertainty. Our choice in this respect is to neglect those IDA's which fluctuate too wildly and treat the remaining approximants on the same ground. To this end we determine the smallest interval that contains half of the results. The values that we shall quote in the tables below as our best estimates correspond to the centre of this interval, the first number in brackets gives the half of the size of the interval. The second number in brackets gives the error induced by the error of

$\beta_c = 0.2216544(3)$ and of the critical index used for biasing. Together they give an idea of the uncertainty of the estimate. As we shall see, in the range of coupling in which we are interested, the uncertainty will be always dominated by the spread of IDA's.

3 The simulations

3.1 Simulations in the low temperature phase.

We simulated the Ising spin model in its low temperature phase at $\beta = 0.2391$, 0.23142, 0.2275, 0.2260,0.2240 and 0.22311 using a demon-algorithm implemented in the multispin coding technique. A detailed discussion of this algorithm is given in ref. [30]. The update of a single spin takes 46×10^{-9} sec. on a HP 735 and 21×10^{-9} sec. on a DEC Alpha 250 workstation for $L = 120$. For $\beta = 0.22311$ the integrated autocorrelation time of the magnetization was $\tau_{int} = 81.(2.)$ in units of sweeps.

We used cubical lattices with periodic boundary conditions and a linear extension of about 20ξ. It should be noted that test runs revealed that in contrast to the high temperature phase a linear lattice size of 6ξ is clearly not sufficient to obtain results close to the thermodynamic limit. Some informations on the simulations are collected in tab.2.

We computed $\langle S_0 \; S_\tau \rangle$ for all values of τ available on the finite lattice. We evaluated $S_0 \; S_\tau$ for all three lattice directions and all possible translations. The connected part was then obtained by subtracting the expectation value of the square of the magnetization $\langle (1/V \; \sum_n s_n)^2 \rangle$.

In order to obtain an estimate for the true correlation length we started from the ansatz

$$G(\tau) \;\; \propto \;\; \exp(-\tau/\xi) \; + \; \exp(-(L-\tau)/\xi) \quad , \tag{36}$$

where the last term takes into account the periodicity of the lattice. An effective correlation length $\xi_{eff}(\tau)$ is then computed by solving the equation above for τ and $\tau+1$. Ignoring the term $\exp(-(L-\tau)/\xi)$, $\xi_{eff}(\tau)$ takes the form

$$\xi_{eff}(\tau) = \frac{1}{\ln(G(\tau+1)) - \ln(G(\tau))}; \tag{37}$$

For $\tau > 3\xi$ $\xi_{eff}(\tau)$ seems to stabilize within error bars. In tab.3 the results for $\tau = 3\xi$ are given. However it is important to stress that there might still be systematic errors due to higher excitations that are of the same magnitude as the statistical error of ξ_{eff}. Therefore a multi-exponential ansatz might be useful. We shall further discuss this point in sect. 4.4 below.

We computed the second moment of the correlation function by

$$\mu_2 = \sum_{\tau=1}^{\tau_{max}} \tau^2 G'(\tau) + \sum_{\tau=\tau_{max}+1}^{\infty} \tau^2 \quad G'(\tau_{max}) \;\; \exp(-(\tau-\tau_{max})/\xi_{eff}(\tau_{max})) \tag{38}$$

where $G'(\tau) = C_{eff}(\tau) \quad \exp(-\tau/\xi_{eff}(\tau))$. Where again $C_{eff}(\tau)$ and $\xi_{eff}(\tau)$ are obtained from

$$G(\tau) \; \propto \; \exp(-\tau/\xi) \; + \; \exp(-(L-\tau)/\xi) \quad , \tag{39}$$

inserting τ and $\tau+1$. As for the exponential correlation length we used $\tau_{max} = 3\xi$ for the data reported in tab.3 . Note that the systematic error introduced by the finite τ_{max} only affects the second term of eq. 38, which is small compared to the first one. Hence these systematic error can safely be ignored for our choice of τ_{max}. The susceptibility is computed analogously. The results are summarized in tab. 3.

Table 2: *Statistics of the runs in the low temperature phase.*

β	L	measures	sweeps/measure	bits
0.2391	30	40000	25	64
0.23142	40	50000	25	64
0.2275	50	50000	25	64
0.2260	80	50000	25	64
0.2240	100	92000	25	32
0.22311	120	124000	25	32

Table 3: *Results in the low temperature phase*

β	L	m	E	ξ_{exp}	ξ_{2nd}	χ
0.2391	30	0.667162(20)	0.553732(17)	1.2851(28)	1.2335(15)	4.178(3)
0.23142	40	0.570306(16)	0.478046(12)	1.8637(45)	1.8045(21)	9.394(4)
0.2275	50	0.491676(14)	0.430364(10)	2.578(7)	2.5114(31)	18.706(10)
0.2260	80	0.449984(16)	0.409609(4)	3.103(7)	3.0340(32)	27.596(11)
0.2240	100	0.372490(10)	0.378612(3)	4.606(13)	4.509(6)	61.348(34)
0.22311	120	0.320830(10)	0.362946(2)	6.208(18)	6.093(9)	112.60(7)

3.2 Simulations in the high temperature phase.

We simulated the Ising spin model in the high temperature phase using the single cluster algorithm [31]. The time slice correlation function was determined using the cluster improved estimator. Finite size effects are less important in this phase and some preliminary test showed that lattice sizes greater than 6ξ are enough to keep them under control. This is clearly visible in the data of tab.4 where we report a test at $\beta = 0.21931$.

Also the determination of the correlation length in the high temperature phase turns out to be much easier than in the low temperature phase. The effective correlation length approaches a plateau quite quickly and the true correlation length is well approximated by ξ_{eff} at a self-consistently chosen distance $\tau \approx \xi_{eff}$. This

Table 4: *Test for finite size corrections at* $\beta = 0.21931$.

β	L	stat	ξ_{exp}	ξ_{2nd}	E	χ
0.21931	40	50000x1000	8.701(4)	8.691(5)	0.313986(24)	303.23(30)
0.21931	50	50000x1000	8.747(4)	8.741(5)	0.313870(18)	307.14(31)
0.21931	60	50000x1000	8.754(6)	8.750(5)	0.313811(15)	307.79(31)
0.21931	70	50000x1000	8.758(4)	8.751(5)	0.313823(14)	307.95(31)
0.21931	80	50000x1000	8.766(5)	8.760(5)	0.313849(14)	308.58(31)

behaviour of ξ_{eff} also implies that the difference between the second moment correlation length and the true correlation length is much smaller that in the low temperature phase. We have chosen the β values in the high temperature phase such that $\beta_c - \beta = \beta_{low} - \beta_c$, where β_{low} are the inverse temperature used in the simulations of low temperature phase. The reason for this choice will be clear in the following section. The uncertainty of β_c virtually does not affect the following analysis. The simulation results are summarized in tab.5. Recently in ref. [32]

Table 5: *Results in the high temperature phase*

β	L	stat	ξ_{exp}	ξ_{2nd}	E	χ
0.20421	20	50000x500	2.363(1)	2.346(1)	0.262928(49)	25.255(16)
0.21189	30	50000x500	3.477(2)	3.465(2)	0.284663(35)	52.09(4)
0.21581	40	50000x500	4.864(3)	4.854(3)	0.298366(28)	98.90(10)
0.21731	50	50000x800	5.892(3)	5.885(3)	0.304493(20)	143.04(12)
0.21931	80	50000x1000	8.766(5)	8.760(5)	0.313849(14)	308.58(31)
0.22020	100	50000x1200	11.884(9)	11.877(7)	0.318742(11)	557.57(61)

Monte Carlo results for the second moment correlation length and the magnetic susceptibility were reported. Interpolation of their results, using the scaling ansatz, to our β-values leads to results consistent with ours. One has to note however, that our statistical errors are at least 3 times smaller than those of ref. [32] in the common β-range.

4 Analysis of the results

4.1 Magnetization

A very precise Montecarlo study of the behaviour of the magnetization in the Ising model can be found in [14]. In particular, in [14] it was shown that the magnetization values obtained from the montecarlo simulations were well described by the following empirical approximation:

$$m(\beta) = t^{0.32694109}(1.6919045 - 0.34357731\ t^{0.50842026} - 0.42572366\ t) \qquad (40)$$

with t given by eq.(4). Since our values of β are inside the region of validity of this approximation, it is interesting to compare also our magnetization values with eq.(40). Notice as a side remark, that our estimates for the magnetization are in general more precise of those reported in [14] (where however a much larger number of β values was studied). This comparison is reported in tab.6, together with the estimates obtained with a double biased IDA analysis of the series published in [28].

Table 6: *Comparison of our Monte Carlo results for the magnetization with eq. 40 and with double biased IDA's.*

β	our MC	eq. 10 of [14]	biased IDA
0.2391	0.66716(2)	0.667143	0.667151(3)(1)
0.23142	0.570306(16)	0.570279	0.570300(16)(1)
0.2275	0.491676(14)	0.491645	0.49167(4)(1)
0.2260	0.449984(16)	0.449953	0.44999(7)(1)
0.2240	0.372490(10)	0.372471	0.37253(15)(2)
0.22311	0.320830(10)	0.320809	0.3209(2)(1)

It is interesting to notice that our values are always in perfect agreement with those obtained from the series expansions, and that our results become more precise than the strong coupling ones starting from $\xi \sim 2$.

The agreement with the data of ref [14] is also very good. Even if our values are systematically slightly higher than those of eq.(40), they are well inside the error bars reported in tab.1 of ref [14].

4.2 Susceptibility

In tab.7 we report the comparison of our data on the susceptibility in the low temperature phase with a double biased IDA analysis of the series published in [28]. The agreement is again very good and as in the previous case our results become more precise than the strong coupling ones starting from $\xi \sim 2$.

Table 7: *Comparison of our Monte Carlo results for the susceptibility with double biased IDA's.*

β	our MC	biased IDA
0.2391	4.178(3)	4.1801(16)
0.23142	9.394(4)	9.401(20)
0.2275	18.706(10)	18.76(15)
0.2260	27.596(11)	27.67(40)
0.2240	61.348(34)	61.8(2.7)
0.22311	112.60(7)	114.6(10.5)

4.3 Second moment correlation length

In tab.8 we report the comparison of our data on the susceptibility in the low temperature phase with a double biased IDA analysis of the series published in [26]. Also in this case the agreement is very good. In addition we give Montecarlo results of ref. [33] for which the β-value matches with ours. One has to note that the results of ref. [33] were obtained with $L \approx 4.7\xi$ and $L \approx 7.8\xi$ for $\beta = 0.226$ and $\beta = 0.224$ respectively.

Table 8: *Comparison of our Monte Carlo results for the second moment correlation length with double biased IDA's.*

β	our MC	MC of ref [33]	biased IDA
0.2391	1.2335(15)		1.2358(16)
0.23142	1.8045(21)		1.803(5)
0.2275	2.5114(31)		2.509(11)
0.2260	3.0340(32)	3.22(1)	3.034(16)
0.2240	4.509(6)	4.61(6)	4.493(30)
0.22311	6.093(9)		6.084(46)

4.4 Exponential correlation length

As we mentioned above, in the low temperature phase the evaluation of the exponential correlation length is much more delicate than in the high temperature phase. In particular, we know from the fact that the ratio $\frac{\xi}{\xi_{2nd}}$ is significantly different from 1 and from [34] that in this region the spectrum is very rich and that nearby states exist that could contaminate the measure of ξ. This is exactly the situation discussed in sect.3.1 above and accordingly we may expect some systematic error in ξ_{eff}. Since the presence of nearby masses is a rather common situation in the broken symmetry phases of statistical mechanical models and since, notwithstanding this, the estimator ξ_{eff} is commonly used also in this cases, we have decided to devote this section to a detailed analysis of this problem. We can explicitly see that ξ_{eff} evaluated according to eq.(37) is not a good estimator of the true correlation length by comparing our estimates with those of ref. [34] (see tab.9). In ref. [34] we computed the glueball spectrum of the Z_2 gauge theory in 3 dimensions. In $d = 3$ the spin and gauge Ising models are related by duality and the inverse of the 0^+ glueball mass exactly coincides with exponential correlation length of the spin Ising model. In ref. [34] we used a variational approach, using 27 different wilson loops as operators, to obtain a faster convergence of ξ_{eff}. In this way each mass of the spectrum was driven in a different channel, and practically no contamination from higher states was present. The results of [34] are comparable in statistical accuracy with the ones presented here. It is easy to see looking at tab.9 that the values of ξ_{eff} obtained here are systematically smaller, in average by a factor of 0.995(2).

This shows, as expected, that the single exponential ansatz is problematic in this case and that a multi-mass ansatz is needed. In order to have an independent test of this fact we tried to fit our data at $\beta = 0.224$ for the correlation function with the following 3-mass ansatz

$$G(\tau) \quad \sim \quad c_1 \; \exp(-\tau/\xi_1) + c_2 \; \exp(-\tau/\xi_2) + c_3 \; \exp(-\tau/\xi_3) \quad . \tag{41}$$

However, the main problem of such multi-mass fits is that they are in general rather unstable under variation of the fit-range. This was also the case with our fits. Therefore we fixed the values of $\xi_2 = 2.50$ and $\xi_3 = 1.70$ found in ref. [34]. We found the following values: $c_2/c_1 \approx 0.1$ and $c_3/c_1 \approx 0.04$, when τ's in the range of 5 to 25 are included in the fit. Notice however that even in this case the results were still rather unstable, and for this reason we cannot give reliable error bars for our estimates. Assuming that $G(\tau)$ is well described by the three mass-ansatz and using our estimates for c_2/c_1 and c_3/c_1 we obtain $\xi_{eff}(3\xi) = 0.993\xi$ which is indeed consistent with our result above.

It is also interesting to insert our estimate for c_2/c_1 and c_3/c_1 into eq. 24. We obtain $\xi/\xi_{2nd} = 1.024$.

The situation is much simpler in the high temperature phase, where $\xi_{2nd} \sim \xi$, no nearby masses are present and ξ_{eff} is a good estimator of the true correlation length.

Table 9: Comparison of the results for the exponential correlation length with those obtained for the Z_2 gauge model ($\tilde{\beta}$ denotes the dual of β)

$\tilde{\beta}$	β	ξ_{gauge}	ξ_{eff}
0.72484	0.23910	1.296(3)	1.2851(28)
0.74057	0.23142	1.864(5)	1.8637(45)
0.74883	0.22750	2.592(5)	2.578(7)
0.75202	0.22600	3.135(9)	3.103(7)
0.75632	0.22400	4.64(3)	4.606(13)

5 Universal amplitude ratios

The standard approach to evaluate the amplitude ratios is to fit the data obtained for both phases separately with the expected scaling law, and then take the ratio of the amplitudes obtained from the fits.

However the bias introduced by the uncertainty in the critical exponent can be avoided by directly studying the ratio as a function of the reduced temperature. This is the reason for which we carefully chose the couplings so as to have the same differences $\Delta\beta \equiv |\beta - \beta_c|$ in the two phases. To better explain our approach let us study as an example the amplitude ratio Γ_χ. From the data reported in tab.3

and 5 we can compute the ratios of susceptibilities in the low and high temperature phase as a function of $\Delta\beta$.

$$\Gamma_\chi(\Delta\beta) = \frac{\chi(\beta_c - \Delta\beta)}{\chi(\beta_c + \Delta\beta)} \tag{42}$$

As $\Delta\beta$ goes to 0 we expect $\Gamma_\chi(\Delta\beta)$ to converge to the amplitude ratio C_+/C_-. However the approach to this critical value is rather non-trivial. A naive implementation of the scaling hypothesis would suggest that the data thus obtained should be constant within the error, but it is easy to see, by looking at the data in tab.10 that this is not the case. There are in fact two source of corrections. The fact that observables in both phases are involved in the ratio tells us that we must expect a correction proportional to $\Delta\beta$. Moreover we certainly expect a "correction to scaling" contribution proportional to $\Delta\beta^\theta$. In the example of Γ_χ the need of such corrections is clearly evident. Looking at the data in tab.10 we see that the violations of scaling are much larger than our statistical errors. Even for the smallest values of $\Delta\beta$ we see no stabilization of Γ_χ within error bars. Following the above discussion we fitted the data of tab.10 with the law

$$\Gamma_\chi(\Delta\beta) \;=\; C_+/C_- \;+\; a_0\,\Delta\beta^\theta \;+\; a_1\,\Delta\beta \quad . \tag{43}$$

where we assumed that there is no other correction to scaling exponent θ' between θ and 1. The results of these fits for the various ratios in which we are interested are reported in tab.11. The fact that we always find rather low χ^2_{red} strongly supports the correctness of the above assumption.

Table 10: The various ratios as functions of $\Delta\beta$

$\Delta\beta$	Γ_χ	Γ_ξ	u	Γ_c	$\frac{\xi}{\xi_{2nd}}$
0.01745	6.044(5)	1.902(2)	15.00(6)	4.394(9)	1.042(3)
0.00977	5.546(5)	1.920(3)	14.75(5)	3.850(10)	1.033(3)
0.00585	5.283(3)	1.932(3)	14.66(6)	3.577(10)	1.026(4)
0.00435	5.182(5)(1)	1.939(3)	14.64(5)	3.466(8)	1.0227(34)
0.00235	5.027(6)(2)	1.942(3)	14.47(6)	3.308(9)	1.0215(42)
0.00146	4.947(6)(3)	1.948(3)(1)	14.51(7)	3.233(9)(1)	1.0188(45)

Few comments are in order at this point.

a1] When dealing with combinations of observables all in the same phase we don't need a correction to scaling term proportional to $\Delta\beta$. This is the case of the coupling constant u and of the ratio ξ/ξ_{2nd}. In this case we fitted with the law[3]:

$$u(\Delta\beta) \;=\; u^* \;+\; a_0\,\Delta\beta^\theta \tag{44}$$

[3]Notice however that, hidden in the $\Delta\beta$ correction there should also be a term proportional to $\Delta\beta^{2\theta}$ which, due to the fact that $\theta \sim 1/2$ is essentially indistinguishable from $\Delta\beta$. The correction

a2] The error due to the specific choice of β_c is always very small. We have reported its value in the data of tab.10 only when it is not negligible. In these cases the number in the first bracket gives the statistical errors of our data, while the second takes into account the uncertainty of the inverse critical temperature.

a3] In the last row of tab.11 we have reported our final results for the various ratios. The corresponding errors take also into account the uncertainty in the index θ and are thus slightly larger than those extracted by the fits.

a4] In the results reported in tab.11 we always fitted only the last five values of $\Delta\beta$, and systematically discarded the data at $\Delta\beta = 0.01745$[4].

a5] A particular care must be devoted to the study of the ratio $\frac{\xi}{\xi_{2nd}}$. It is possible to prove that if higher masses exist in the theory (and this is the case in both phases of the Ising model) then $\frac{\xi}{\xi_{2nd}}$ must certainly be larger than 1. This is a consequence of eq.(24) and of the fact that the coefficients c_i which appear in it must be positive (see eq. (9) and (10) of ref. [34] for a proof of this last statement).

In the high temperature phase $\frac{\xi}{\xi_{2nd}}$ is almost compatible with 1 (within the errors), and we can only use our data to set an upper bound for its value which, looking at the data with the largest correlation length, can be safely chosen to be $\frac{f_+}{f_{+,2nd}} < 1.0006$.

On the contrary in the $\beta > \beta_c$ phase the quantity $\frac{\xi}{\xi_{2nd}} - 1$ is much larger than the error bars, and can be measured rather precisely. The data in the last column of tab.10 and 11 refer to this case and use ξ_{eff} (defined in sect.4.4) as estimator of ξ. Hence we must add to the result of the fit (which is reported in the last row of tab.11: $\frac{f_-}{f_{-,2nd}} = 1.009(5)$) the contribution due to the systematic underestimation $\Delta\xi \sim 0.007$ discussed in sect.4.4 above. Taking into account also this correction we quote as our final result $\frac{f_-}{f_{-,2nd}} = 1.017(7)$. It is interesting to notice that this result agrees within the errors with the value $\frac{f_-}{f_{-,2nd}} \sim 1.024$ obtained in sect. 4.4 by inserting in eq.(24) our estimates for c_2/c_1 and c_3/c_1 and the values of the two nearby (inverse) masses ξ_2 and

$\Delta\beta^{2\theta}$ should be present also in the case in which all the observables belong to the same phase, thus suggesting to use also in this case the fit eq.(43). It turns out however that such a $\Delta\beta^{2\theta}$ correction, if present, has a negligible amplitude and for this reason we confined ourselves to the fit (44).

[4]This is not due to the fact that adding this data we had a poorer fit, on the contrary we checked that that for all the ratios the fit keeping *all the six* data was always equally good. The reason of our choice is that we tried to confine ourselves to the narrowest possible region near the critical point compatible with a reasonable precision for the results. This allows us to trust in our assumption of neglecting other possible, unknown, corrections to scaling which are certainly present but hopefully negligible in this range. It is a remarkable consequence of the high precision of our montecarlo estimates that we can still extract meaningful results by using only five values of $\Delta\beta$.

ξ_3 extracted from ref. [34]. Finally we can directly estimate the ratio by using the unbiased data for ξ obtained in ref. [34] and reported in tab. 9. The only problem is that these data are slightly less precise and that the lowest value of β is missing. The resulting estimate for the ratio: $\frac{f_-}{f_{-,2nd}} = 1.029(11)$ is thus affected by a larger error. Also in this case we find a good agreement within the errors with our final result $\frac{f_-}{f_{-,2nd}} = 1.017(7)$

Table 11: Results of the fits according to eq.(43) and (44) (see also the comment (a5) above).

	Γ_χ	Γ_ξ	u	Γ_c	$\frac{\xi}{\xi_{2nd}}$
χ^2_{red}	0.52	0.43	0.52	0.01	0.20
C.L.	59%(2)	65%	67%	98%	90%
a_0	3.5(8)	-0.04(50)	4.8(1.2)	3.0(1.5)	0.24(8)
a_1	46.9(5.4)	-2.9(3.4)		54(10)	
final result	4.75(3)	1.95(2)	14.3(1)	3.05(5)	1.009(5)

5.1 Comparison with other existing estimates

In tab.14 we have compared our results with those obtained with other methods. Let us briefly comment on this comparison:

b1] There are three possible approaches to the evaluation of the amplitude ratios. Montecarlo simulations ("MC" in tab.14), low and high temperature series expansions ("HT,LT", in tab.14) and field theoretic methods. In this last case two different approaches are possible. The first one consists in looking at the ϵ expansion of the ϕ^4 theory around four dimensions ("ϵ-exp." in tab.14). The second one consists in looking directly to the ϕ^4 theory in three dimensions ("d=3" in tab.14). For a detailed discussion of these approaches see for instance ref. [35]. Let us also mention for completeness that an independent, interesting method to evaluate the ratio of specific heat amplitudes (which we do not study in this paper) by looking at the distribution of the zeroes of the partition function was proposed and applied to the Ising model in [36, 37].

b2] The results of ref. [2, 4] for Γ_χ and Γ_ξ were obtained with a careful resummation of two loop ϵ-expansions. On the contrary, the ϵ-expansion for the exponential ξ (which is needed to obtain the value of $f_-/f_{-,2nd}$ which is reported in the last column of tab.14) is known only at one loop, hence the value $f_-/f_{-,2nd} \sim 1.005$ of [2] must only be considered as indicative. Later the ϵ-expansion for Γ_χ was extended up to ϵ^3 and the value of [4] corresponds to the Padè resummation of such series. Recently in [5] this result has been further improved by using the parametric representation of the equation of state of the theory.

b3] The $d = 3$ approach originates from a suggestion of Parisi [17]. While the results of [5, 6] make use only of series expansions obtained in the symmetric phase of the theory, in [7, 8] a three loop calculation, directly performed in the low temperature phase, was used. In this last case a crucial role is played by the low temperature renormalized coupling constant u evaluated at the critical point. We shall further comment on this point later.

b4] The estimates for the amplitude ratios obtained with the use of low and high temperature series expansion reported in tab.14 are mainly taken from ref [9]. They were obtained with the use of IDA's on the series reported in tab.12, where we used the standard notations: $v \equiv th(\beta)$ and $u \equiv e^{-4\beta}$ (not to be confused with the low temperature coupling constant in the $d = 3$ ϕ^4 theory!) for high and low temperature series respectively. Recently these data have been reanalyzed in [10] (using the same series) leading to essentially the same results. Notice however that in these last years new longer series have been constructed in the low temperature phase. These are the series that we have used in the previous sections to test our montecarlo results. It would be very interesting to see if these new series can lead to improved estimates of the amplitude ratios. In particular it would possible now to analyze also the ratio ξ/ξ_{2nd} which was previously unaccessible, since the two series for ξ and for ξ_{2nd} start to be different only at the order u^6

Table 12: *Some informations on the series used in ref.[9]*

ref.	year	observable	length
[38]	1979	HT/ χ	v^{18}
[39]	1969	HT/ ξ_{2nd}	v^{12}
[41]	1979	LT/ χ	u^{20}
[40]	1975	LT/ ξ_{2nd}	u^{15}
[40]	1975	LT/ ξ	u^7
[41]	1979	LT/ m	u^{21}

Table 13: *Some informations on the low temperature series used in this paper.*

ref.	year	observable	length
[28]	1993	LT/ χ	u^{32}
[26]	1995	LT/ ξ_{2nd}	u^{23}
[27]	1995	LT/ ξ	u^{15}
[28]	1993	LT/ m	u^{32}

b5] Some of the data reported in tab.14 (those which are underlined) have been obtained combining separate amplitudes reported by the authors, thus their errors are most probably overestimated.

b6] It has been recently reported in [43] an estimate for the ratio $\frac{f_+}{f_{+,2nd}}$ obtained with a strong coupling expansion to 15th order of the correlation function $G(\tau)$. The result is $\frac{f_+}{f_{+,2nd}} = 1.00023(5)$ which agrees with our bounds $1. < \frac{f_+}{f_{+,2nd}} < 1.0006$.

Table 14: *Results for the amplitude ratios reported in the literature*

ref.	year	method	$\frac{C_+}{C_-}$	$\frac{f_{+,2nd}}{f_{-,2nd}}$	u^*	$\frac{C_+}{f_{+,2nd}^3 B^2}$	$\frac{f_-}{f_{-,2nd}}$
[2, 3]	1974	ϵ-exp	~ 4.8	~ 1.91			~ 1.005
[4]	1985	ϵ-exp	~ 4.9				
[5]	1996	ϵ-exp	4.70(10)				
[6]	1987	$d=3$	4.77(30)			3.02(8)	
[5]	1996	$d=3$	4.82(10)				
[8]	1996	$d=3$	4.72(17)	2.013(28)	15.1(1.3)		
[9]	1989	HT,LT	4.95(15)	1.96(1)	14.8(1.0)	3.09(8)	
[42]	1993	HT,LT			14.73(14)		
[11]	1994	MC	5.18(33)	2.06(1)	17.1(1.9)	3.36(23)	
This work	1996	MC	4.75(3)	1.95(2)	14.3(1)	3.05(5)	1.017(7)

5.2 Comparison with an effective potential model.

It has been recently proposed to study the critical properties of the three dimensional Ising model by constructing the effective potential of the corresponding quantum field theory. This effective potential is constructed simulating the model for various values of the external magnetic field. This program was carried on in [44] for the high temperature phase of the model and was recently extended to the broken symmetric phase in [45]. The main result is that in the effective potential, besides the expected ϕ^4 term also a ϕ^6 term is present. It is interesting to test this model with our high precision results. Fortunately one of the values of β studied in [45]: $\beta = 0.2260$ exactly coincide with one of our values thus allowing a detailed comparison. This comparison is reported in tab.15, where in the third column we have reported the values of the various observables directly measured at $\beta = 0.2260$ (tab.1 of [45]) while in the last column we have reported the same observables obtained from what was considered in [45] as the most successful fitting procedure (tab.2 of [45]). Notice that χ is the inverse of V'' and that $u = 3G$ in the notations of [45]. Our results are in general one order of magnitude more precise than those of ref. [45]. It is interesting to see that both m and χ are in rather good agreement with our data. The only strong disagreement is in the value of ξ_{2nd} and, as a consequence of this, in u. Most likely this disagreement is only due to the too small lattices studied in ref. [45] (the lattice size is reported in the last line of tab.15) and does not imply that the approach proposed in [45] is wrong.

Table 15: *Comparison with Tsypin's results.*

Observable	This work	Tab.1 of [45]	Tab.2 of [45]
m	0.449984(16)	0.44975(17)	0.44975(17)
χ	27.596(11)	27.397(75)	27.586(198)
ξ_{2nd}	3.0340(32)	2.946(6)	2.956(10)
u	14.64(5)	15.90(9)	15.84(28)
L	80	30	30

5.3 Comparison with experimental data

The experimental data reported in tab.16 refer to the three most important experimental realization of the Ising universality class, namely binary mixtures (bm), liquid-vapour transitions (lvt) and uniaxial antiferromagnetic systems (af). It is important to notice that these realizations are not on the same ground. Antiferromagnetic systems are particularly apt to measure the C_+/C_- and $f_{+,2nd}/f_{-,2nd}$ ratios, while for the liquid vapor transitions the $\Gamma_c \equiv R_c/R_\xi^3$ combination is more easily accessible. Finally, in the case of binary mixture all the three ratios can be rather easily evaluated. Even if obtained with very different experimental setups all these estimates qualitatively agree among them and this is certainly one of the most remarkable experimental evidences of universality. When looking in more detail at the various results one can see a residual small spread among them (even if in general the various estimates are compatible within the quoted experimental uncertainties). This spread is mainly due to the presence of correction to scaling terms whose amplitudes vary as the experimental realizations are changed and that are difficult to control. Thus some care is needed to compare these experimental data with theoretical estimates. The common attitude is to assume that the above systematic errors are randomly distributed and to take the weighted mean of the various experimental results.

Table 16: *Experimental estimates for some amplitude ratios*

exp. setup	$\frac{C_+}{C_-}$	$\frac{f_{+,2nd}}{f_{-,2nd}}$	$\frac{C_+}{f_{+,2nd}^3 B^2}$
(bm)	4.4(4)	1.93(7)	3.01(50)
(lvt)	4.9(2)		2.83(31)
(af)	5.1(6)	1.92(15)	
(all of them)	4.86(46)	1.93(12)	2.93(41)
This work	4.75(3)	1.95(2)	3.05(5)

Following this line we have reported in tab.16 the weighted means (together with, in parenthesis, the standard deviations), of the experimental results reported in [1]. In the first three rows we have studied separately the three different realizations of the universality class while in the fourth row all the experimental data at disposal

are analyzed together. In the last row we have reported our results.

In the case of the $f_{+,2nd}/f_{-,2nd}$ ratio (for which, as we have seen, some of the present theoretical or montecarlo estimates disagree) we have listed, for a more detailed comparison, all the available experimental data in tab.17. In this table we denote with "N-H" the nitrobenzene – n-hexane binary mixture, and with "I-W" the one obtained by mixing isobutyric acid and water. A much more detailed account of the various experimental estimates can be found in [1].

Table 17: *Experimental estimates for the* $\frac{f_{+,2nd}}{f_{-,2nd}}$ *ratio*

Ref.	year	exp. setup	$\frac{f_{+,2nd}}{f_{-,2nd}}$
[46]	1971	(af),MnF_2	1.7(3)
[47]	1972	(af),FeF_2	2.06(20)
[48]	1980	(af),CoF_2	1.93(10)
[49]	1983	(bm),N-H	1.9(2)
[50]	1986	(bm),I-W	2.0(4)
This work	1996	MC	1.95(2)

6 Conclusions

We have estimated various universal amplitude ratios in the case of the three dimensional Ising model. Our final results are:

$$\frac{C_+}{C_-} = 4.75(3) \qquad \frac{f_{+,2nd}}{f_{-,2nd}} = 1.95(2) \qquad \frac{f_-}{f_{-,2nd}} = 1.017(7) \tag{45}$$

$$u^* \equiv \frac{3\ C_-}{f^3_{-,2nd} B^2} = 14.3(1) \qquad \frac{C_+}{f^3_{+,2nd} B^2} = 3.05(5) \quad . \tag{46}$$

Our results are in general in good agreement with other estimates of the same quantities obtained with field theoretical methods or with high/low temperature series. The main discrepancy that we have found is with the Montecarlo results of ref. [11] and with some of the results of ref. [45]. It must also be noticed that our result $\frac{f_{+,2nd}}{f_{-,2nd}} = 1.95(2)$ only marginally agrees with that of [8]: 2.013(28). However, as we mentioned above, this result depends on the value of the coupling constant u^* which is an external input in the calculations of [8]. By plugging our value of u^* in the perturbative expansion of [8] we find a lowering of the ratio with respect to [8]. This lower result: $\frac{f_{+,2nd}}{f_{-,2nd}} = 1.99(2)$ agrees not only with the strong/weak coupling result [9], but also with our estimate. Finally it is important to notice that our results are also in reasonable agreement with the experimental ones.

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